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On the nonergodic dynamics of the Ising anti-Hebbian model

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Abstract
We study the nonergodic dynamics of the Ising spin anti-Hebbian model. The mean-field method for Glauber dynamics is reviewed in our context. The resulting one-site problem is described by the memory and noise term, which are expressed by the large-scale convolutions of correlation and response functions. These memory and noise terms are governed by the relation similar to the generalized fluctuation–dissipation relation of nonergodic dynamics. The results of replica method are recovered by studying the onset of very slow dynamics.

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1. Introduction

Glassy systems are attractive subjects in the recent study of statistical physics [1, 2]. Since the problem is related to the nonergodicity of the system, the theory should be based on the dynamical equation [3], which is far more difficult than the study based on the Gibbs measure. Recently, the nonergodic dynamics of the continuous spin model, such as the spherical p-spin (SPS) model, was directly studied by Langevin equation and gives quite interesting and suggestive results [4]. Especially, it was clarified that the correlation and response functions satisfy a remarkable relation, which is called generalized fluctuation–dissipation (GFD) relation [5–7]. Thanks to this relation, the study of the dynamics is simplified greatly. On the other hand, replica method, which is based on the Gibbs measure, gives fruitful idea on the property of complicated energy landscape [8]. These two approaches often give similar results in the mean-field theory, although they start with different formulations. Some authors suggested that there is a deep relation between these two approaches apart from mean-field theory [9].

In the study of statistical physics, Ising spin models have been very important. With discrete spin variables, the stochastic dynamics should be controlled by Glauber dynamics [10].

In spite of the expected generality of nonergodic dynamics, the situation of such models is not so clear, although some special spin models were studied by mean-field method based on Glauber dynamics [5]. We are especially interested in the dynamics of the infinite range Ising spin models with two-body interactions, which have been studied intensively by replica method and simulations [11–13]. Interestingly, in the framework of replica theory, it was suggested that dynamical phase transition can be identified by studying the marginally stable one-step replica symmetry breaking (RSB) solution. This seems to describe the numerical simulations correctly. Recently, this idea was applied to the anti-Hebbian (AH) model [14] and long-range anti-ferromagnetic spin models to identify the dynamical phase transition [15, 16]. The AH model is defined by reversing the sign of interactions of the Hopfield model. For some parameter region, the AH model has glassy low-temperature phase, which is quite different from the spin-glass state of the Sherrigton–Kirkpatrick (SK) model [17]. This difference is formally characterized by the different behaviour of the solutions of replica method. We expect that the property of replica solution will provide good references in the study of dynamics of these models.

The purpose of this paper is to study the mean-field method for the Glauber dynamics of the AH model at low temperature. Main interest is on the structure of the effective one-site problem characterized by two-body interactions, which will induce rather involved memory and noise terms in the effective one-site problem. We will study how the GFD relation works to find the result which corresponds to replica theory. We also discuss the large time-scale property of the self-consistent equation by assuming a simple form on the correlation and response functions. For Ising Glauber dynamics, we mainly follow the formulation presented in [5].

This paper is organized as follows. In section 2, we review the Glauber dynamics of the Ising spin models and evaluate the quenched averages over interactions, which give a one-site problem with memory and noise terms. In section 3, we discuss the solutions for slow dynamics and discuss the relation with replica method. We also discuss the fluctuation–dissipation (FD) relation for the memory and noise terms, which plays a very important role in our calculations. In section 4, we discuss the long time dynamics with some speculations. Section 5 is devoted to some discussions.

2. Basic formulation

This section is devoted to the description of the AH model, glauber dynamics and mean-field method, following the standard formulation.

2.1. Glauber dynamics of Ising spin models

The infinite range Ising spin models are defined by N Ising spin $\sigma_i = \pm 1$ ($i = 1, 2, \ldots, N$, where N is a system size) and interactions J_{ij}. The energy function is given by

$$E = -\frac{1}{2}\sum_{i\neq j} J_{ij}\sigma_i\sigma_j - \sum_i \eta_i\sigma_i, \tag{2.1}$$

where η_i are external fields on site i. For the AH model, the interactions are given by

$$J_{ij} = -\frac{1}{N}\sum_{\mu=1}^{P} \xi_i^\mu \xi_j^\mu, \tag{2.2}$$

where ξ_i^μ are quenched random variables which take ± 1 with probability $1/2$. Note the minus sign in (2.2). For $\alpha = P/N$ smaller than 1.4, replica method implies that this model has a dynamical phase transition. We can also see that the model reduces to the SK model in the large α limit by studying the correlation among J_{ij}. We will see this later.

Glauber dynamics is a stochastic process in which the probability of spin configuration, $\{\sigma_i\}$, $P\{\sigma_i\}$, obeys the dynamical equation given by

$$\frac{\mathrm{d}P\{\sigma_i\}}{\mathrm{d}t} = \sum_{\{\sigma_i'\}} W(\{\sigma_i\}, \{\sigma_i'\}) P\{\sigma_i'\} \tag{2.3}$$

where

$$W(\{\sigma_i\}, \{\sigma_i'\}) = \sum_i w_i(\sigma_i, \sigma_i') \tag{2.4}$$

The matrices $w_i(\sigma_i, \sigma_i')$ control the spin flip on site i. It is a function of energy change caused by the spin flip $\sigma_i' \to \sigma_i$. In Ising spin models, the energy changes are given by the twice of local fields $h_i' = \eta_i(t) + \sum_j J_{ij}\sigma_j'$, where $\eta_i(t)$ are assumed to be time dependent. The details of $w_i(\sigma_i, \sigma_i')$ will not matter much as long as it gives the thermal equilibrium states described by Gibbs–Boltzmann distribution at least with time-independent $\eta_i(t)$ and in high-temperature. The typical form of the matrix will be

$$w_i(\sigma_i, \sigma_i') = \tfrac{1}{2}(-\sigma_i\sigma_i')\{1 - \sigma_i' \tanh(\beta h_i')\} \tag{2.5}$$

where $\beta = T^{-1}$ is an inverse temperature.

To discuss the evolution equations, it is convenient to introduce the quantum-mechanics-like formulation [18], which suggests

$$w_i(t) = \sum_{\sigma_i, \sigma_i'} |\sigma_i\rangle w_i(\sigma_i, \sigma_i') \langle \sigma_i'|, \tag{2.6}$$

where $|\sigma_i\rangle$ are orthogonal state vectors. For the whole system, we introduce direct products of one-site vectors and define $W(t) = \sum_i w_i(t)$. Then, the formal solution of (2.3) is given by the time-ordered product of $(1 + \mathrm{d}t\, W(t))$. Introducing the state-vector representation $|P(t)\rangle$ for the whole system, we have

$$|P(t)\rangle = U(t, 0)|P(0)\rangle \tag{2.7}$$

where

$$U(t'', t') = T \exp \int_{t'}^{t''} W(t)\, \mathrm{d}t, \tag{2.8}$$

where T means a time-ordered product. The correlation and response functions are defined by

$$C_i(t, t') = \langle \sigma_i(t)\sigma_i(t')\rangle$$
$$G_i(t, t') = \frac{\partial\langle\sigma_i(t)\rangle}{\partial\eta_i(t')}$$

where

$$\langle\cdots\rangle = \langle 1|T \cdots U(\infty, 0)|P(0)\rangle$$

where $\langle 1|$ means a vector with equal weight for all component. Time variable t in $\sigma_i(t)$ simply indicates the place where σ_i should be located.

For the thermal equilibrium state, $w_i(t)|P(t)\rangle = 0$, we can show the fluctuation–dissipation relation

$$G_i(t, t') = \beta\theta(t - t') \frac{\partial C_i(t, t')}{\partial t'} \tag{2.9}$$

by using $\partial_t\sigma_i(t) = \sigma_i(t)w_i(t) - w_i(t)\sigma_i(t)$ and $\partial w_i(t)/\partial\eta_i(t) = -\beta(w_i(t)\sigma_i(t) + \tanh(\beta h_i)w_i(t))$. Since relation (2.9) is independent of the realization of interactions, it should hold after quenched average over J_{ij} at least in the paramagnetic phase.

2.2. Averages over quenched random interactions

This subsection is devoted to the derivation of the mean-field one-site problem. The details are found in appendix A. We first rewrite the evolution equation in the form

$$U(t,0) = \int U(t,0,\{h\})H\{h\}\prod_{t',i} \mathrm{d}h_i(t') \tag{2.10}$$

where the distribution function of local fields $H\{h\}$ is defined by

$$H\{h\} = \prod_{t,i} \delta(h_i(t) - \eta_i(t) - \sum_{j\neq i} J_{ij}\sigma_j(t)) \tag{2.11}$$

where $\delta(x)$ is a delta function. Denoting the averages over quenched interactions by $[\cdots]$ and introducing the integral representations for delta functions [19], we have

$$[H\{h\}]\prod_{t',i}\mathrm{d}h_i(t') = \int D\{h,\bar{h}\}\left[\exp\sum_{i<j} J_{ij}\Omega_{ij}\right]\exp\left(-\sum_i\int_0^t \bar{h}_i(t')(h_i(t') - \eta_i(t'))\,\mathrm{d}t'\right)$$

where $D\{h,\bar{h}\} = \prod_{t,i}\mathrm{d}h_i(t)\,\mathrm{d}\bar{h}_i(t)/2\pi\sqrt{-1}$, with $\sqrt{-1}$ being an imaginary unit, and

$$\Omega_{ij} = \int_0^t (\sigma_i(t')\bar{h}_j(t') + \sigma_j(t')\bar{h}_i(t'))\,\mathrm{d}t'. \tag{2.12}$$

The average over interactions can be performed by expanding the exponential in terms of interactions as was done for the replica method [14]. Then, we obtain

$$\left[\exp\sum_{i<j} J_{ij}\Omega_{ij}\right] = \exp\frac{1}{2}N\,\mathrm{Tr}\,A\left(\frac{\Omega}{N}\right). \tag{2.13}$$

The form of $A(x)$ depends on the correlation among interactions. For the SK model, $A_{\mathrm{SK}}(x) = x^2/2$, whereas for the AH model, it is given by $A(x) = -\alpha\{\ln(1+x) - x\}$. In the following argument, we use the expression $A(x) = \sum_n a_n x^n$, where $a_n = (-1)^n/n$ for the AH model. Note $A(x)$ reduces to $A_{\mathrm{SK}}(x)$ for $\alpha \to \infty$ after replacing $x \to x/\sqrt{\alpha}$.

In the mean-field theory, we should find the one-site problem with mean fields, which are expressed by the time correlation function and response function

$$C(t,t') = \frac{1}{N}\sum_i \langle\sigma_i(t)\sigma_i(t')\rangle, \tag{2.14}$$

$$G(t,t') = \frac{1}{N}\sum_i \langle\sigma_i(t)\bar{h}_i(t')\rangle, \tag{2.15}$$

while $\langle\bar{h}_i(t)\bar{h}_i(t')\rangle = 0$, which preserves the normalization of probability as discussed in [3]. Actually, this expression is obtained by differentiating $\langle 1|P(\infty)\rangle = 1$ with respect to $\eta_i(t)$ and $\eta_i(t')$. Similarly, by using partial integrals, we see that $G(t,t')$ equals the response function introduced before. Thus, $G(t,t') = 0$ for $t < t'$.

We should find the one-site problem in the action by replacing the variables with the same site indices according to (2.14) and (2.15). We describe the derivation in appendix A. Due to $\langle\bar{h}_i(t)\bar{h}_i(t')\rangle = 0$ and $G(t<t') = 0$, almost all terms in $\mathrm{Tr}(\Omega/N)^n$ disappear, leaving

$$\frac{1}{2}Na_n\,\mathrm{Tr}\left(\frac{\Omega}{N}\right)^n \sim \frac{1}{2}na_n\sum_i\left\{\int \bar{h}_i(t'')D_{n-1}(t'',t')\bar{h}_i(t')\,\mathrm{d}t''\,\mathrm{d}t' + 2\int \bar{h}_i(t'')\Sigma_{n-1}(t'',t')\sigma_i(t')\,\mathrm{d}t''\,\mathrm{d}t'\right\}$$

where

$$D_{n-1}(t'',t') = \sum_{k=0}^{n-2} \int G^{n-2-k}(t'',t_1)C(t_1,t_2)G^k(t',t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$\Sigma_{n-1}(t'',t') = G^{n-1}(t'',t')$$

where

$$G^k(t'',t') \equiv \int \cdots \int G(t'',t_1)\cdots G(t_{k-1},t')\,\mathrm{d}t_1 \cdots \mathrm{d}t_{k-1}$$

$G^0(t,t')$ should be regarded as $\delta(t-t')$. Since $G(t,t') = 0$ for $t < t'$, integral time variables in $G^k(t'',t')$ are of descending order. Note that, in D_{n-1}, the earliest time variables in the convolutions of G, t_1 and t_2 are identical to the time variables in $C(t_1,t_2)$.

Putting these expressions in $H\{h\}$ and introducing one-site state vector $|p(t)\rangle$. we obtain the one-site problem

$$|p(t)\rangle = \int D\{h,\bar{h}\}T\exp\{L(t,0)\}|p(0)\rangle, \tag{2.16}$$

where

$$L(t,0) = \int_0^t w(t')\,\mathrm{d}t' - \int_0^t \bar{h}(t')(h(t') - \eta(t'))\,\mathrm{d}t' + \frac{1}{2}\int_0^t\int_0^t \bar{h}(t'')D(t'',t')\bar{h}(t')\,\mathrm{d}t''\,\mathrm{d}t'$$
$$+ \int_0^t\int_0^t \bar{h}(t'')\Sigma(t'',t')\sigma(t')\,\mathrm{d}t''\,\mathrm{d}t'$$

and

$$D(t'',t') = \sum_{n=2}^{\infty} na_n D_{n-1}(t'',t') \tag{2.17}$$

$$\Sigma(t'',t') = \sum_{n=2}^{\infty} na_n \Sigma_{n-1}(t'',t'). \tag{2.18}$$

This completes the reduction to one-site problem.

$\Sigma(t,t')$ gives a memory effect and $D(t,t')$ gives a correlation of effective Gaussian noises. For our model, $\Sigma(t,t')$ is made of convolutions of response function of the surrounding spins. This property will be quite general for two-body interactions. Note that they are given simply by correlation and response functions themselves for the SK model and usual products of correlation and response functions for the p-spin model. As discussed in the next section, we can show that $D_{n-1}(t,t')$ and $\Sigma_{n-1}(t,t')$ satisfy the fluctuation–dissipation relation if $G(t,t')$ and $C(t,t')$ do.

To conclude this section, we present the energy expectation value in terms of mean fields. For this purpose, it is sufficient to express $Ne = [E(t)\exp\,\mathrm{Tr}\,J\Omega/2]$ in terms of mean fields, where $E(t)$ is the energy function with spin variables $\sigma_i(t)$ with very large t and $\eta_i(t) = 0$. The details are presented in appendix A. By the expansion in terms of interactions and replacing the site-paired variables by correlation and response functions, we obtain

$$e = -\frac{1}{2}\int_0^t (G(t,t')D(t,t') + C(t,t')\Sigma(t,t'))\,\mathrm{d}t'. \tag{2.19}$$

Note that the diagrams that are not connected to $E(t)$ do not contribute to Ne because $\langle\sigma(t)\bar{h}(t')\rangle = 0$ for $t < t'$ and $\langle\bar{h}(t)\bar{h}(t')\rangle = 0$.

3. Self-consistent equations

In the previous section, we have obtained an effective one-site spin dynamics with memory and noise terms. In this section, we discuss the self-consistent equations based on several assumptions. The relation between dynamical theory and replica theory is also discussed.

First, we review the thermodynamics of the effective one-site action in the paramagnetic phase. According to [5], if functions $D(t, t')$ and $\Sigma(t, t')$ in the effective one-site action (2.17) satisfy the relation

$$\Sigma(t, t') = \beta\theta(t - t')\frac{\partial D(t, t')}{\partial t'}, \tag{3.1}$$

there exists a thermal equilibrium state which is described by Gibbs–Boltzmann weight and the fluctuation–dissipation relation

$$\langle\sigma(t)\bar{h}(t')\rangle = \beta\theta(t - t')\frac{\partial\langle\sigma(t)\sigma(t')\rangle}{\partial t'} \tag{3.2}$$

holds. We will not review the proof here and only point out that the problem before random average actually has thermal equilibrium state at least in high temperature. We remark that (3.1) holds if $G(t, t')$ and $C(t, t')$ satisfy the FD relation. In appendix C, we show this without assuming the time translation invariance.

In the low-temperature phase, it is natural to assume that spontaneous magnetization appears. Then, the correlation function will be separated into a fast part and a time-independent part. The action is also divided into two parts. The fast part gives the action similar to that of paramagnetic phase. The time-independent part gives rise to time-independent effective fields. Thus, even in the low-temperature phase, the argument above is applicable with suitable modifications. However, this picture oversimplifies the situation of the low-temperature phase. As we will see in the next subsection, the time-independent part cannot be determined self-consistently for small α. This corresponds to the fact that there is no replica symmetry (RS) solution for small α. Accordingly, the time-independent part should be replaced by a slow part, which controls the very slow dynamics in low-temperature phase.

In this section, we first discuss the simplest picture of spontaneous magnetization and then study the slow dynamics.

3.1. The simplest approximation

It is known that RSB solution properly describes the property of spin-glass states for the SK model. However, the simple RS solution identifies the phase transition point correctly. This solution corresponds to the dynamical picture where there is a thermal equilibrium state to which the system relaxed in a thermal time scale τ_0 and stays in this state forever. We first discuss this situation, which will provide some calculus with the fast part and the slow part.

Let us assume

$$C(t, t') = C_f(t - t') + q \tag{3.3}$$

$$G(t, t') = G_f(t - t') \tag{3.4}$$

where $q = \overline{\langle\sigma\rangle^2}$ is a square of spontaneous magnetization, i.e. Edward–Anderson order parameter [20]. $\langle\cdots\rangle$ means a thermal average and $\overline{\cdots}$ means an average over quenched randomness, which is realized by an average over effective local field. $C_f(t - t')$ and $G_f(t - t')$ are the fast part of the functions, which satisfy the FD relation

$G_f(t-t') = \beta\theta(t)\partial C_f(t-t')/\partial t'$. $C_f(t-t')$ varies from $1-q$ to zero in the thermal time scale. Thus, we have

$$\int_0^t G_f(t-t')\,\mathrm{d}t' = \beta(1-q)$$

for $t \gg \tau_0$. Substituting (3.3) and (3.4) into (2.17) and (2.18), we obtain an action in which all correlation and response functions are replaced by the fast-part functions plus an additional term proportional to q, that is

$$L = L_f + \frac{1}{2}\lambda\left(\int \bar{h}(t)\,\mathrm{d}t\right)^2. \tag{3.5}$$

The constant λ in the last terms reduces to

$$\begin{aligned}\lambda &= q\sum_n n a_n \sum_{k=0}^{n-2}\int_0^\infty G_f^{n-2-k}(t)\,\mathrm{d}t\int_0^\infty G_f^k(t')\,\mathrm{d}t' \\ &= q\sum_n n(n-1)a_n\{\beta(1-q)\}^{n-2} \\ &= qA''(\beta(1-q)).\end{aligned}$$

By introducing a Gaussian variable x which obeys the distribution $Dx = \exp(-x^2/2)\,\mathrm{d}x/\sqrt{2\pi}$ in (2.16), the second term in (3.5) induces a time-independent field $\sqrt{\lambda}x$. The rest parts of action drive the system to a thermal equilibrium state. Thus, we obtain

$$\langle\sigma\rangle = \tanh(\beta\sqrt{\lambda}x) \tag{3.6}$$

and self-consistent equation for $q = \overline{\langle\sigma\rangle^2}$ is given by

$$q = \int \tanh^2(\beta\sqrt{\lambda}x)\,\mathrm{D}x \tag{3.7}$$

This is a typical saddle-point equation for RS solution. For the SK model, $\lambda = q$ and the equation predicts the transition temperature correctly, although this solution is known to be unstable as de Almeida–Thouless instability [21]. In the context of dynamics, the solution is expected to be unstable against the slow change of correlation function for $t - t' \gg \tau_0$.

Let us study the RS solution for the AH model [14], for which $na_n = (-1)^n$, which gives

$$\lambda = \frac{\alpha q}{(1+\beta(1-q))^2}. \tag{3.8}$$

Interestingly, (3.7) does not have a solution with $q \neq 0$ for $0 < \alpha < 1$ down to $T = 0$. Formally, this is due to the positive sign of β in λ, which is certainly due to the negative sign of interactions. Roughly speaking, the memory effect exists, as we will see in the following subsections, but is not strong enough to give time-independent effective field. For large enough α, the AH model becomes similar to the SK model. This crossover occurs at $\alpha \sim 1.4$. In this paper, we restrict ourselves to $0 < \alpha < 1$ and discuss the possible slow dynamics.

3.2. Effective slow dynamics

In the previous section, we have assumed that $C(t,t')$ becomes constant q in the thermal time scale $t - t' \sim \tau_0$, and beyond this time scale, $C(t,t')$ sticks to this constant. This corresponds to the assumption that there is one thermal equilibrium state. For the AH model with small α, we cannot find the solution of this property. Even with this situation, the simulation shows that there is a temperature below which the energy no longer obeys the high-temperature expansion. By using replica method, this temperature can be identified by marginally stable RSB solution.

Dynamical interpretation of RSB is that there is a hierarchy of time scales which describes the nonerogdic dynamics. The shortest time scale is the one which characterizes the local thermal equilibrium, while larger time scales describe the transitions among erogdic components. Accordingly, it was proposed that the FD relation should be generalized to be

$$G(t,t') = \beta X(C)\theta(t-t')\frac{\partial C(t,t')}{\partial t'} \tag{3.9}$$

where $X(C)$ is a function of $C(t,t')$ (for recent review, see [1, 2]).

Let us concentrate on the simplest situation. We assume that $C(t,t')$ becomes q after a thermal relaxation time scale $t - t' \sim \tau_0$ and sticks to this value until a large time τ_1 passes. After τ_1, the system gets away from an initial local equilibrium state with $C(t,t')$ decreasing to 0. τ_1 characterizes the onset of nonergodic time scale. Accordingly, we assume that $X(C) = 1$ for $q < C(t,t') \leqslant 1$ and $X(C) = m$ for $0 \leqslant C(t,t') \leqslant q$, where q and m are determined by the self-consistent equation.

Following this picture, we divide the correlation and response functions into the fast part and the slow part.

$$C(t,t') = C_f(t-t') + C_s(t,t'), \tag{3.10}$$

$$G(t,t') = G_f(t-t') + G_s(t,t'), \tag{3.11}$$

$C_f(t-t')$ changes from $1-q$ to 0 in the thermal time scale and $C_s(t,t')$ changes from q to 0 in nonergodic time scale. The assumption on $X(C)$ implies $G_f(t) = -\beta\theta(t)\partial C_f(t)/\partial t$ for the first part and $G_s(t,t') = \beta m\theta(t-t')\partial C_s(t,t')/\partial t'$ for the slow part. Because of the slow change of $C_s(t,t')$, $G_s(t,t')$ is expected to be very small, although the integral of $G_s(t,t')$ is not negligible.

Correspondingly, $D(t,t')$ and $\Sigma(t,t')$ are divided into the fast part and the slow part by substituting the expressions above. The fast parts of $D(t,t')$ and $\Sigma(t,t)$ are defined simply by replacing all $C(t,t')$ and $G(t,t')$ by $C_f(t-t')$ and $G_f(t-t')$ in each function. They are denoted by $D_f(t-t')$ and $\Sigma_f(t-t')$. Then, the slow parts $D_s(t,t')$ and $\Sigma_s(t,t')$ are defined by

$$D(t,t') = D_f(t-t') + D_s(t,t')$$

$$\Sigma(t,t') = \Sigma_f(t-t') + \Sigma_s(t,t').$$

As discussed in appendix B, these equations mean that the slow part contains many terms which are convolutions of the fast part and the slow part of correlation and response functions. Interestingly even in this situation, the GFD relation between $D_s(t,t')$ and $\Sigma_s(t,t')$ holds as discussed in appendix C. Thus, we can write

$$\Sigma(t,t') = \beta X(C)\theta(t-t')\frac{\partial D(t,t')}{\partial t'}. \tag{3.12}$$

We expect that the time dependence of $D(t,t')$ is similar to $C(t,t')$. Let us denote $D(t,t)$ by D_0 and $D_s(t,t)$ by D_1, which are assumed to be independent of t. Then, $D_f(t-t')$ varies from $D_0 - D_1$ to zero in the thermal time scale τ_0. On the other hand, $D_s(t,t')$ sticks to D_1 in $t - t' \sim \tau_1$ and varies from D_1 to zero in the nonergodic time scale. Further, $D_s(t,0)$ tends to zero as $t \to \infty$ if $C(t,0)$ tends to zero, as shown in appendix B. We will express D_0 and D_1 in terms of q and m later.

Using the separation of $D(t,t')$ and $\Sigma(t,t')$, the action is divided into the fast part and the slow part:

$$L = L_f + L_s, \tag{3.13}$$

where

$$L_f = \int w(t')\,\mathrm{d}t' - \int \bar{h}(t')(h(t') - h_s(t'))\,\mathrm{d}t' + \frac{1}{2}\iint \bar{h}(t'')D_f(t''-t')\bar{h}(t')\,\mathrm{d}t''\,\mathrm{d}t' + \iint \bar{h}(t'')\Sigma_f(t''-t')\sigma(t')\,\mathrm{d}t''\,\mathrm{d}t'$$

$$L_s = -\int \bar{h}(t')h_s(t')\,\mathrm{d}t' + \frac{1}{2}\iint \bar{h}(t'')D_s(t'',t')\bar{h}(t')\,\mathrm{d}t''\,\mathrm{d}t' + \iint \bar{h}(t'')\Sigma_s(t'',t')\sigma(t')\,\mathrm{d}t''\,\mathrm{d}t'$$

where $h_s(t)$ is a slow part of $h(t)$ and does not vary in the time scale smaller than τ_1. $\eta(t)$ is dropped for simplicity. L_f brings a thermal equilibrium state characterized by local field $h_s(t)$. In the time scale $t - t' < \tau_1$, $D_s(t,t')$ and $\Sigma_s(t,t')$ can be regarded as constants. Accordingly, $\bar{h}(t)$ in L_s are replaced by an average over τ_1. Similarly, we can replace $\sigma(t)$ in L_s by the average over a time scale τ_1, which is given by $\phi(t) = \langle\sigma(t)\rangle = \tanh(\beta h_s(t))$. In this way, we obtain the effective slow action given by

$$L_s = -\int \bar{h}(t')h_s(t')\,\mathrm{d}t' + \frac{1}{2}\iint \bar{h}(t'')D_s(t'',t')\bar{h}(t')\,\mathrm{d}t''\,\mathrm{d}t' + \iint \bar{h}(t'')\Sigma_s(t'',t')\phi(t')\,\mathrm{d}t''\,\mathrm{d}t'.$$

Time variables in this expression should be replaced by t/τ_1. However, to avoid the complexity of notation, we use the same time variables for L_s as L_f. This will not cause trouble as long as we are careful when the fast part and the slow part coexist.

3.3. Correspondence with replica method

Having an effective slow action, we first address the onset condition of the slow dynamics, which will turn out to be the saddle-point equation and marginally stable condition of the one-step RSB ansatz [5]. To establish a correspondence with replica theory, we write the self-consistent equations for the typical values of the functions $C_s(t,t')$ and $G_s(t,t')$. For this purpose, we need to find the distribution function for $h_s(t)$. As discussed in appendix D, dynamical equation gives a distribution function of $x = h_s/\sqrt{D_1}$, which reads

$$p(x)\,\mathrm{d}x = \frac{1}{z}\cosh^m(\beta\sqrt{D_1}x)\,\mathrm{D}x \tag{3.14}$$

where $D_1 = D_s(t,t)$ and z is a normalization constant. Note that m in this expression comes from the GFD relation. Then, the self-consistent equation for $q = \overline{\langle\sigma\rangle^2} = C_s(t,t)$ reads

$$q = \int \tanh^2(\beta\sqrt{D_1}x)\,p(x)\,\mathrm{d}x \tag{3.15}$$

which is identical to the saddle-point equation of the one-step RSB solution, if D_1 is given suitably.

We need another equation to determine m. Onset of glass dynamics is signalled by non-zero $G_s(t,t')$ with $t' \sim t$. In the next section, we obtain the dynamical equation for $G_s(t,t')$, which reduces to

$$G_s(t,t') \sim \overline{\chi(t)\chi(t')}\Sigma_s(t,t')$$

for small $t - t'$, where $\chi(t) = \beta(1 - \phi^2(t))$. Further, $\Sigma_s(t,t')$ is proportional to $G_s(t,t')$ for small $t - t'$ which is still large enough to make an approximation

$$G_f^k(t_1 - t_2) \sim (\beta(1-q))^k\delta(t_1 - t_2)$$

for the fast part. Using this expression, we obtain

$$\Sigma_s(t,t') \sim \sum_n n(n-1)a_n\{\beta(1-q)\}^{n-2}G_s(t,t').$$
$$= G_s(t,t')A''(\beta(1-q)).$$

Then, we obtain

$$1 = A''(\beta(1-q))\int \chi^2 p(x)\,\mathrm{d}x \tag{3.16}$$

where $\chi = \beta(1-\tanh^2(\beta\sqrt{D_1}x))$. This is certainly the marginally stable condition obtained by replica theory.

Remaining job is to express $D_1 \equiv D_s(t,t)$ in terms of q and m. With large enough $t \gg \tau_1$, integrating the relation $\Sigma(t,t') = \beta X(C)\theta(t-t')\partial D(t,t')/\partial t'$ over t', we obtain

$$\int_{t-\tau_1}^{t} \Sigma(t,t')\,\mathrm{d}t' = \beta(D_0 - D_1),$$
$$\int_0^t \Sigma(t,t')\,\mathrm{d}t' = \beta(D_0 - D_1 + mD_1)$$

where we used $D_s(t,0) = 0$. On the other hand, the integrals on the left-hand sides are given by the definition of $\Sigma(t,t')$ and the GFD relation between $C(t,t')$ and $G(t,t')$, giving $A'(\beta(1-q))$ and $A'(\beta(1-q+mq))$, respectively. Thus, we have

$$D_0 = \frac{1}{m}A'(\beta x_m) + \left(1-\frac{1}{m}\right)A'(\beta x_0)$$
$$D_1 = \frac{1}{m}(A'(\beta x_m) - A'(\beta x_0))$$

where $x_0 = 1-q$ and $x_m = 1-q+mq$. Putting these results in the self-consistent equations, we recover the results obtained by the replica theory [8, 14].

Repeating the same procedure, we can obtain the energy expectation value e as follows:

$$e = -\frac{1}{2}\beta\int_0^t X(C)\partial_{t'}(C(t,t')D(t,t'))\,\mathrm{d}t'$$
$$= -\frac{1}{2}\beta(D_0 - qD_1 + mqD_1)$$
$$= -\frac{1}{2}\left\{\frac{1}{m}x_m A'(\beta x_m) + \left(1-\frac{1}{m}\right)x_0 A'(\beta x_0)\right\}$$

which also equals the replica result.

To be complete, we briefly describe the solution of the self-consistent equations (3.15) and (3.16). In the replica theory of the AH model with small α, there is no RS solution, but there are two kinds of RSB solutions, one is the static solution which is obtained by extremizing the free energy with respect to q and the block size m of non-zero replica order parameter. This solution is stable and describes the absolute minimum state. Other is the dynamical solution which is defined by the marginally stable condition of the replicon modes. We found that this condition is identical to onset condition of nonergodic dynamics as described in this section. Actually, for $0 < \alpha < 1.4$, (3.15) and (3.16) have a solution, which gives $q \sim 1$ and $m \sim 1$ near the transition temperature. For small α, the transition temperature of the dynamical solution becomes much higher than that of static one. The annealing simulations are consistent with the dynamical transition given by dynamical solution.

4. Dynamical equation for long time scale

Having the same results as the replica theory, we address the dynamics of the time scale larger than τ_1, where the correlation function decreases to zero. Although the study in this section is far from conclusive, we find several interesting aspects of the self-consistent equation.

4.1. Effective dynamical equation and onset of nonergodic dynamics

By introducing Gaussian noise $\xi(t)$ in L_s, we write

$$\int \exp(L_s)\,\mathrm{d}h\,\mathrm{d}\bar{h} \propto \int \exp K\{\xi(t)\} \prod_t \delta(h_e(t) - h(t)) \prod_t \mathrm{d}h(t)\,\mathrm{d}\xi(t)$$

where the effective field $h_e(t)$ is given by

$$h_e(t) = \int_0^t \Sigma_s(t, t')\phi(t')\,\mathrm{d}t' + \xi(t) \tag{4.1}$$

and $K\{\xi(t)\}$ defines the Gaussian distribution of $\xi(t)$ with $\overline{\xi(t)\xi(t')} = D_s(t, t')$, where $\overline{\cdots}$ in this section means an average over $\xi(t)$. Then, we write the dynamical equation

$$\phi(t) = \tanh(\beta h_e(t)), \tag{4.2}$$

$\phi(t)$ directly depends on $\xi(t)$ and also depends on $\xi(t')$ with $t' < t$ through $\phi(t'')$ with $t' < t'' < t$. The correlation and response functions are given by $\xi(t)$-average of $\phi(t)\phi(t')$ and $\delta\phi(t)/\delta\xi(t')$. Using the expression for $\phi(t)$, we have

$$\frac{\delta\phi(t)}{\delta\xi(t')} = \chi(t)\delta(t - t') + \chi(t)\int_{t'}^t \Sigma_s(t, t'')\frac{\delta\phi(t'')}{\delta\xi(t')}\,\mathrm{d}t'' \tag{4.3}$$

where $\chi(t) = \beta(1 - \phi^2(t))$, which equals the time integral of fast time response function. Note that the fast functions work as a delta function in the long time scale. Thus we should assume $G(t, t') = \chi(t)\delta(t - t') + G_s(t, t')$, implying

$$\left(\frac{\delta\phi(t)}{\delta\xi(t')}\right)_s = \chi(t)\Sigma_s(t, t')\chi(t') + \chi(t)\int_{t'}^t \Sigma_s(t, t'')\left(\frac{\delta\phi(t'')}{\delta\xi(t')}\right)_s \mathrm{d}t''. \tag{4.4}$$

This is the basic equation in the study of long time dynamics. There is no delta-function-like contribution in the integral since all quantities are made of slow parts. Thus the integral becomes zero for $t' \to t$, which implies $G_s(t, t') \sim \overline{\chi(t)\chi(t')}\Sigma_s(t, t')$ in this limit. This is the result used in the previous section. Using (4.4), we can write $G_s(t, t')$ in terms of $\chi(t)$ and $\Sigma_s(t, t')$. Explicitly,

$$G_s(t, t') = \sum_n \overline{\chi(\Sigma_s\chi)^n(t, t')}. \tag{4.5}$$

With the GFD relation, this makes the self-consistent equation for correlation and response functions. Formally, $\chi(t)$ are functions of Gaussian variables $\xi(t)$ whose correlation is given by $D_s(t, t')$. However, it will not be easy to express the correlation among $\chi(t)$ in terms of the mean fields. In the next section, we study some approximation to evaluate the right-hand side of (4.5).

4.2. Study of nonergodic dynamics

By rather general argument, it was suggested that the correlation function $C_s(t, t')$ is generally expressed by the function $C(\varphi(t')/\varphi(t))$ [7], where $\varphi(t)$ is an increasing function of t. This expression violates the time translation invariance, which reflects the effect of ageing. The solution for the SPS model is given by the simple case $C(x) = qx$, which gives the long time correlation and response functions

$$C_0(t,t') = q_0 \frac{\varphi(t')}{\varphi(t)} \tag{4.6}$$

$$G_0(t,t') = g_0 \frac{\varphi'(t')}{\varphi(t)} \tag{4.7}$$

for $t > t'$, where $g_0 = \beta m q_0$ due to the GFD relation. As we will see, the above form has a very good property as correlation and response functions, just like an exponential function. Following calculation is somewhat tentative and far from conclusive but its simplicity may be helpful in gaining some insight into the calculus of slow dynamics.

Let us assume that $C_s(t,t') = C_0(t,t')$ and $G_s(t,t') = G_0(t,t')$ for the AH model as a working assumption. We may think that, for a very long time scale for which $C_s(t,t') \sim 0$, the function $C(x)$ will be approximated to be the first order of x. With this approximation, the convolutions of $G_0(t,t')$ are easily evaluated as presented in appendix E. Interestingly, for the AH model, $\Sigma_s(t,t')$ is given by a very concise form

$$\Sigma_0(t,t') = \sigma_0 \frac{\varphi^b(t')}{\varphi^b(t)} \frac{\varphi'(t')}{\varphi(t)} \tag{4.8}$$

where $b = a g_0$ with $a = (1 + \beta(1-q))^{-1}$ and $\sigma_0 = \alpha a^2 g_0$. Note that the above form is true only for the AH model. The explicit form of $\Sigma_0(t,t')$ depends on the interactions. Since $b > 0$, $\Sigma_0(t,t')$ decreases faster than $G_0(t,t')$. This is because the correlation of interactions alternately changes sign as the number of interactions increases due to frustration.

Now we need some approximations to evaluate the right-hand side of (4.5). Since $\chi(t) > 0$, we simply assume

$$\overline{\prod_{i=1}^{n} \chi(t_i)} = \overline{\chi}^n \tag{4.9}$$

where $\overline{\chi} = \beta(1-q)$. Since this gives the lower bound of the right-hand side of (4.5), we denote the approximated expression by $G_L(t,t')$. Using this expression, we have

$$G_L(t,t') = \sigma_0 \overline{\chi}^2 x^{-k} x^b \frac{\varphi'(t')}{\varphi(t)} \tag{4.10}$$

where $x = \varphi(t')/\varphi(t)$ and $k = \overline{\chi}\sigma_0$. Note the negative sign in front of k, which is due to the positive weights of the summation. The self-consistency would be achieved if $\alpha a^2 \overline{\chi}^2 = 1$ and $b = k$, which is $1 = \alpha a \overline{\chi}$. These two equations require $a\overline{\chi} = 1$, which are not satisfied since $a\overline{\chi} < 1$. We also note that the first equation becomes the marginality condition if $\overline{\chi}^2$ is replaced by $\overline{\chi^2}$. This may imply that the correlation among $\chi(t)$ is important to obtain meaningful result.

In addition to the correlation among $\chi(t)$, we remark on the effect by short time terms in the convolution. Let us discuss the correction term to C_0 such as $(\varphi(t')/\varphi(t))^2$. Although this term itself decreases faster than $\varphi(t')/\varphi(t)$, it will give a logarithmic contribution in the convolutions, which will lead to the change of exponent. We should note that the factor $a = (1 + \beta(1-q))^{-1}$ in b, which arises from a fast thermal time scale, contributes to the exponent in Σ_0.

The study in this section may imply an important aspect of glassy dynamics. That is, with large-scale convolutions, the short time property of correlation functions plays an important role in determining the large time property of the solution. This situation looks strange when compared with the second-order phase transition. It will be fruitful to study how to take into account the correlation among $\chi(t)$ and how to find corrections terms in C_s and G_s.

5. Discussion

In this paper, we have studied the nonergodic dynamics of the Ising AH model by the mean-field method. The resulting one-site problem has memory and noise terms which are given by the large-scale convolutions of correlation and response functions. These convolutions naturally represent a chain of effects of other spins. The action is divided into the fast part and the slow part in the same way as the SPS model. As expected, memory and noise terms are governed by the fluctuation–dissipation relation $\Sigma(t,t') = \beta X(C)\theta(t-t')\partial D(t,t')/\partial t'$ in the same way as correlation and response functions. Thanks to this relation, the seemingly involved one-site action is reduced to rather simple form in large time scale.

The dynamics of the AH model with small α is quite different from that of the SK model. In the AH model with small α, there is no time-independent field determined self-consistently, which corresponds to the absence of the RS solution. Consequently, we need to assume that the correlation function decreases to zero in the very long time scale. This implies that the low-temperature states are not similar to some time-independent states but similar to the snapshot of a paramagnetic state. The thermal time scale is denoted by τ_0, while onset of nonergodic time scale is denoted by τ_1. Onset of the slow part gives the equation identical to the marginally stable condition of the replica method. By this study, we found that m in the GFD relation is identical to the block size of the replica order parameter. This relation has been found in several mean-field models [4, 5] and confirmed by rather general argument [9].

To study the dynamics in nonergodic time scale, we have derived the dynamical equation for $G_s(t,t')$, which consists of $\Sigma_s(t,t')$ and $\chi(t)$. To discuss the self-consistent equation, we adopted the assumption $G_0(t,t') = g_0\varphi'(t')/\varphi(t)$ for $G_s(t,t')$. We found that the calculus with $G_0(t,t')$ shows some interesting aspects, which looks like a natural extension of the exponential function. Disregarding the correlation among $\chi(t)$, the self-consistent equation is evaluated explicitly, although this does not achieve the self-consistency. In this calculation, the function $G_0(t,t')$ shows an interesting property under convolution, which looks similar to the exponential function, such as $\exp(ct')/\exp(ct)$. This aspect is quite appealing and we expect that the correct form of $C(x)$ will be obtained by small change from qx.

Although the AH model looks quite special, we found that this model is very similar to the long-range anti-ferromagnets (AF), which are defined by spatially decreasing anti-ferromagnetic interactions. The energy function of these models is expressed by the summation of quadratic constraint terms on spin variables. This implies that these models belong to the same category of glassy system. Actually, by introducing a replica method without random averages, we showed that these models have similar low-temperature glassy states [15, 16], which is also confirmed by numerical simulations. Following the spirit of a replica method without random averages [15], we may formulate the dynamical method to study the long-range anti-ferromagnetic spin models, which will give self-consistent equations similar to those of the AH model. Since there are various anti-ferromagnetic or competitive interactions in nature, the study in this direction will hopefully lead to a deeper understanding of nonequilibrium phenomena in nature.

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Appendix A

In this appendix, we review the diagrammatic calculations of mean-field one-site action and energy expectation value.

The first step is to do ξ_i^μ-average of $H\{h\}$. This is performed in the same way as the replica method [14]. Since $J_{ij} \sim 1/\sqrt{N}$, we expand the action in terms of J_{ij}, yielding

$$\exp \sum_{i<j} J_{ij}\Omega_{ij} = \prod_{i<j} \left\{ 1 + J_{ij}\Omega_{ij} + \frac{1}{2}(J_{ij}\Omega_{ij})^2 + \cdots \right\} \tag{A.1}$$

where

$$\Omega_{ij} = \int (\bar{h}_j(t)\sigma_i(t) + \bar{h}_i(t)\sigma_j(t))\,\mathrm{d}t. \tag{A.2}$$

Factors $J_{ij}\Omega_{ij}$ are diagrammatically represented by lines with ends at i and j. For the AH model, ξ_i^μ-average is performed by using the expectation values

$$\overline{J_{ij}J_{jk}\cdots J_{ki}} = \frac{(-1)^n\alpha}{N^{n-1}} \tag{A.3}$$

where J_{ij} makes a loop of length n with all different site indices. In the diagrams which are made of several loops, each loop gives the contribution given by (A.3). In this way, ξ_i^μ-average of (A.1) is expressed by products of loops. After exponentiating and summing over sites, they reduce to

$$N\alpha \sum_{n=2}^{\infty} \frac{(-1)^n}{2n} \mathrm{Tr} \left(\frac{\Omega}{N}\right)^n. \tag{A.4}$$

Note we need a factor $1/2n$ to count the number of the diagrams in $\mathrm{Tr}(\Omega/N)^n$ correctly.

Let us study the structure of $\mathrm{Tr}(\Omega/N)^n$. The term $\bar{h}_i\sigma_j$ can be expressed by an arrow starting from site i and ending at site j. Then, $\mathrm{Tr}\,\Omega^n$ contains 2^n diagrams with possible orientations of arrows. To do the mean-field approximation, it is convenient to group them according to the number of $\bar{h}_j(t)\bar{h}_j(t')$. The diagrams without this factor have all arrows in the same orientation. Then they are expressed by the product of $\sum_i \sigma_i(t)\bar{h}_i(t')/N$. The diagrams with one $\bar{h}_j\bar{h}_j$ should have one $\sigma_i\sigma_i$ pair. The diagrams with more than one $\bar{h}_j\bar{h}_j$ will not contribute in the mean-field approximation due to $\langle\bar{h}_j\bar{h}_j\rangle = 0$. After site sum, the action is expressed by the convolutions of the form $\sum_i a_i(t)b_i(t')/N$, where $a_i(t)$ and $b_i(t)$ are either $\bar{h}_i(t)$ or $\sigma_i(t)$.

Having the convolutions of the form $\sum_i a_ib_i/N$, one-site problem is obtained as follows. For this form, we introduce mean-field functions $F = G$ or C depending on the pair a_i and b_i. By substituting $\sum_i a_ib_i/N = F + \delta F$, where $\delta F = \sum_i a_ib_i/N - F$, and keeping the first-order terms of δF in the action, dynamical variables are completely decoupled, giving an effective one-site problem. The self-consistent equation is certainly given by $\langle\delta F\rangle = 0$, where $\langle\cdots\rangle$ means an average by the one-site problem.

Let us discuss the effective one-site action on site 0. As discussed above, the diagrams should be grouped according to the number of $\bar{h}_j(t)\bar{h}_j(t')$ pair. We first note that the diagrams with more than one $\bar{h}_j(t)\bar{h}_j(t')$ pair do not contribute to the one-site action due to the relation $\langle\bar{h}_j(t)\bar{h}_j(t')\rangle = 0$. If there is no such pair, diagrams have arrows with the same orientation, which implies that mean-field functions are all G. Then, the contribution to the one-site problem is given by

$$2\bar{h}_0(t)G^{n-1}\sigma_0(t') \tag{A.5}$$

after site summation. If there is one $\bar{h}_j\bar{h}_j$ pair, site j should be identical to site 0. If it is not, we have zero mean field on site 0 due to $\langle\bar{h}_j(t)\bar{h}_j(t')\rangle = 0$. With $\bar{h}_0\bar{h}_0$, there is one $\sigma_j\sigma_j$ pair in the diagram, which gives rise to C. Then, we obtain the expression

$$\bar{h}_0(t)G^{n-2-k}CG^k\bar{h}_0(t'), \tag{A.6}$$

where G^k means a suitable convolution of response functions. k takes $0, 1, 2, \ldots, n-2$ depending on the place of $\sigma_j\sigma_j$ pair. We need a factor 1/2 to avoid double counting for both expressions. Recovering time variables in C and G and summing over n, we find the one-site action presented in section 2.

Now let us discuss the energy expectation value defined by

$$Ne = -\left[\sum_{k<l} J_{kl}\sigma_k(t)\sigma_l(t)\exp\left(\sum_{i<j} J_{ij}\Omega_{ij}\right)\right]. \tag{A.7}$$

Let us fix k and l for time being. In the same way as above, the contributing diagrams coming from an exponential factor are expressed by the chains of arrows from k to l. Among them, the diagrams with arrows with the same direction starting from l and ending at k give ΣC_k. The contribution of the rest of the diagrams is given by DG_k. We need a factor 1/2 to avoid double counting. Summing over site indices, we have

$$e = -\frac{1}{2}\int_0^t (C(t,t')\Sigma(t,t') + G(t,t')D(t,t'))\,\mathrm{d}t'. \tag{A.8}$$

Appendix B

In this appendix, we discuss the separation of the fast part and the slow part of $D(t,t')$ and $\Sigma(t,t')$. Symbolically, the slow parts are defined by

$$\begin{aligned} D_s(t,t') &= D\{C_f + C_s, G_f + G_s\} - D\{C_f, G_f\} \\ \Sigma_s(t,t') &= \Sigma\{G_f + G_s\} - \Sigma\{G_f\}. \end{aligned} \tag{B.1}$$

Assuming $t > t'$, let us study the expression for $\Sigma_{n-1}(t,t')$. Substituting $G_f + G_s$ and expanding in terms of G_s, we obtain

$$\Sigma_{s,n-1}(t,t') = \sum_{k_1+k_2=n-2} G_f^{k_1}G_sG_f^{k_2}(t,t') + \sum_{k_1+k_2+k_3=n-3} G_f^{k_1}G_sG_f^{k_2}G_sG_f^{k_3}(t,t') + \cdots.$$

In this expression, the products are convolutions and time variables are of descending order. For $t_1 - t_2 \gg \tau_0$, $G_f^{k_1}(t_1,t_2)$ can be replaced by $b^{k_1}\delta(t_1-t_2)$, where $b \equiv \beta(1-q)$ since G_s can be regarded as a constant. We can do the same thing for D_s and obtain

$$\Sigma_{s,n-1}(t,t') = \sum_{p=1}^{n-1}\binom{n-1}{p} b^{n-1-p}G_s^p(t,t')$$

$$D_{s,n-1}(t,t') = \sum_{p=1}^{n-1}\binom{n-1}{p} b^{n-1-p}\sum_{k1+k_2=p-1} G_s^{k_1}C_sG_s^{k_2}(t,t').$$

To obtain the expression for $D_{s,n-1}(t,t')$, we substitute the expression of $\Sigma_{s,n-1}(t,t')$ and use the equality

$$\sum_{k_2\leqslant l\leqslant k-k_1}\binom{k-l}{k_1}\binom{l}{k_2}=\binom{k+1}{k_1+k_2+1}.$$

Strictly, C_s in the last line should be C except for $p = 1$ term, but the difference is negligible in the time scale $t - t' \sim \tau_1$. Note that there is a correspondence between $\Sigma_{s,n-1}(t,t')$ and $D_{s,n-1}(t,t')$ order by order of b.

Let us estimate the value $D_{n-1}(t,0)$ in the large t limit. It is enough to show this for each term in $D_{n-1}(t,0)$, which is explicitly given by

$$G_s^{k_1}C_sG_s^{k_2}(t,t')=\iint G_s^{k_1}(t,t_1)C_s(t_1,t_2)G_s^{k_2}(t',t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2. \tag{B.2}$$

When $t' \sim 0$, we can set $t_2 \sim 0$ since $t' > t_2$. Then, the region of t_1 which contributes to the integral should satisfy $C_s(t_1,0) > 0$ and $\partial C_s(t,t_1)/\partial t_1 > 0$. Although the integral of $G_s^{k_1}(t,t_1)$ can be finite, the factor $C_s(t_1,0)$ becomes very small for large t_1. Thus, the region with finite contribution vanishes and the integral tends to zero for $t \to \infty$.

By using $G = G_f + G_s$, we note an interesting relation

$$\frac{G}{1+G}(t,t')=\frac{G_f}{1+G_f}(t,t')+\frac{a^2G_s}{1+aG_s}(t,t') \tag{B.3}$$

for the time scale $t - t' > \tau_1$, where $a = (1+\beta(1-q))^{-1}$. The first term works like a delta function for the time scale larger than τ_1.

Appendix C

In this appendix, we show the fluctuation–dissipation-like relation between $\Sigma_n(t,t')$ and $D_n(t,t')$. For simplicity, we restrict ourselves to nonergodic time scale, for which $X(C) = m$. All function in this appendix are regarded as slow parts. As discussed in the previous appendix, slow parts of these functions are expressed by the summation of convolutions given by

$$\bar{\Sigma}_p(t,t')=G_s^p(t,t') \tag{C.1}$$

$$\bar{D}_p(t,t')=\sum_{k=0}^{p-1}\iint G_s^{p-k-1}(t,t'')C_s(t'',t''')G_s^k(t',t''')\,\mathrm{d}t''\,\mathrm{d}t'''. \tag{C.2}$$

Since $\Sigma_s(t,t')$ and $D_s(t,t')$ are expressed by these convolutions with the same coefficients, it is enough to show the relation for each p, that is

$$\bar{\Sigma}_p(t,t')=\beta m\theta(t-t')\partial_{t'}\bar{D}_p(t,t') \tag{C.3}$$

by assuming $G_s(t,t') = \beta m\theta(t-t')\partial_{t'}C_s(t,t')$.

For simplicity, the subscript s is dropped in the following expressions. For $p = 1$, the relation is trivial. Let us study the situation for $p = 2$. The functions are expressed as an integral over one time variable, denoted by t''. We assume $t > t'$. Then,

$$\bar{\Sigma}_2(t,t')=\int_{t'}^{t}G(t,t'')G(t'',t')\,\mathrm{d}t'', \tag{C.4}$$

$$\bar{D}_2(t,t')=\int_0^{t'}C(t,t'')G(t',t'')\,\mathrm{d}t''+\int_0^{t}G(t,t'')C(t',t'')\,\mathrm{d}t''. \tag{C.5}$$

Then interval in the last integral $[0-t]$ is divided into $[0-t']$ and $[t'-t]$. By integral by parts for $[0-t']$, we have

$$\bar{D}_2(t,t') = \int_{t'}^{t} G(t,t'')C(t'',t')\,\mathrm{d}t'' + \beta m C(t,t')C(t',t'). \tag{C.6}$$

We assume that $C(t',t')$ is independent of t. By differentiating $\bar{D}_2(t,t')$ with respect to t', we obtain the desired relation for $p=2$. To study the relation for $p \geqslant 3$, it is crucial to divide the integral region into $t > t'$ and $t < t'$ for the integrals which contain $C(t,t')$. To show the relation for general p, we may use the mathematical reduction, that is, by noting the relation $\bar{D}_p = \bar{D}_{p-1}G + \bar{\Sigma}_{p-1}C$ and $\bar{\Sigma}_p = \bar{\Sigma}_{p-1}G$, where products are understood as convolution, we repeat the argument for $p=2$.

Appendix D

In this appendix, following [5], we discuss the distribution of $h_s(t)$ for the effective slow action. Let us write the action in the form

$$\exp\{L(t_1)\} = \int \exp\{L(t_1,t_0) + L(t_0)\} \prod_{t_0 \leqslant t < t_1} \mathrm{d}h(t) \tag{D.1}$$

and demand that $L(t_0)$ and $L(t_1)$ are the same functions of the effective field. Then, $L(t)$ reflects the stationary distribution of effective field in locally equilibrium states. $L(t_1,t_0)$ is the action of the region $t_0 < t < t_1$ defined by

$$\begin{aligned} L(t_1,t_0) = &-\int_{t_0}^{t_1} \bar{h}(t)h(t)\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{t_0}^{t} \bar{h}(t)D_s(t,t')\bar{h}(t')\,\mathrm{d}t\,\mathrm{d}t' \\ &+ \int_{t_0}^{t_1}\int_{t_0}^{t} \bar{h}(t)\Sigma_s(t,t')\phi(t')\,\mathrm{d}t\,\mathrm{d}t' + \int_{t_0}^{t_1} \bar{h}(t)D_s(t,t_0)\eta(t_0)\,\mathrm{d}t \end{aligned} \tag{D.2}$$

where the last term reflects the effect from the region $0 < t < t_0$. We assume the relation $\Sigma_s(t,t') = \beta m\theta(t-t')\partial D_s(t,t')/\partial t'$. The crucial point in the following argument is that we can set $\bar{h}(t) = \beta m\partial_t\phi(t)$ under the integral over $h(t)$. To see this, by introducing

$$\eta(t) = \int^{t} \bar{h}(t')\,\mathrm{d}t' \tag{D.3}$$

we write

$$\begin{aligned} &\int_{t_0}^{t} \bar{h}(t)D_s(t,t')\bar{h}(t')\,\mathrm{d}t' + \int_{t_0}^{t} \bar{h}(t)\Sigma_s(t,t')\phi(t')\,\mathrm{d}t' \\ &\qquad = \bar{h}(t)D_s(t,t)\eta(t) - \bar{h}(t)D_s(t,t_0)\eta(t_0) + \int_{t_0}^{t} \bar{h}(t)\Sigma_s(t,t')y(t')\,\mathrm{d}t' \end{aligned} \tag{D.4}$$

where $y(t)$ is defined by $\eta(t) = \beta m(\phi(t) + y(t))$. Then, integral over t of this expression gives

$$\frac{1}{2}D_s(t,t)(\eta^2(t_1) - \eta^2(t_0)) - \int_{t_0}^{t_1} \bar{h}(t)D_s(t,t_0)\eta(t_0)\,\mathrm{d}t + \text{term proportional to } y(t)$$

where $D_s(t,t)$ is assumed to be independent of t. On the other hand,

$$\begin{aligned} -\int_{t_0}^{t_1} \bar{h}(t)h(t)\,\mathrm{d}t = &-h(t_1)\eta(t_1) + h(t_0)\eta(t_0) + \beta m(\Phi(h(t_1)) \\ &- \Phi(h(t_0))) + \beta m(y(t_1)h(t_1) - y(t_0)h(t_0)) - \beta m \int \frac{\mathrm{d}y(t)}{\mathrm{d}t}h(t)\,\mathrm{d}t \end{aligned} \tag{D.5}$$

where

$$\Phi(h(t)) = \int \phi(t)\frac{\mathrm{d}h}{\mathrm{d}t}\,\mathrm{d}t. \tag{D.6}$$

Note $\Phi(h) = \ln\cosh(\beta h)/\beta$ for $\phi(t) = \tanh(\beta h(t))$. In these expressions, $h(t)$ with $t_0 < t < t_1$ disappears except the last term of (D.5). Now we assume

$$L(t_0) = \tfrac{1}{2}D_s(t_0, t_0)\eta^2(t_0) - h(t_0)\eta(t_0) + \beta m\Phi(h(t_0)). \tag{D.7}$$

Then, $h(t_0)$-dependence in $L(t_1, t_0) + L(t_0)$ disappears except $y(t_0)h(t_0)$ which gives $y(t_0) = 0$ by the integral over $h(t_0)$. Similarly, we have $\mathrm{d}y(t)/\mathrm{d}t = 0$ by the integral over $h(t)$ with $t_0 < t < t_1$. Thus, we have $y(t) = 0$.

Collecting all terms, we find the same $h(t_1)$-dependence for $L(t_1)$ as $L(t_0)$. The expression $L(t)$ implies that the effective field distribution is given by

$$p(x) = \frac{1}{z}\exp\left(-\frac{1}{2}x^2 + \beta m\Phi(\sqrt{D_s(t,t)}x)\right) \tag{D.8}$$

where $x = h/\sqrt{D_s(t,t)}$ and z is a normalization constant.

The argument in this appendix implies that the relation $\langle\phi(t)\bar{h}(t')\rangle = \beta m\langle\phi(t)\partial_{t'}\phi(t')\rangle$ holds under the integral over $h(t)$.

Appendix E

This appendix is devoted to some calculus of the function

$$G_0(t,t') = g_0\theta(t-t')\frac{\varphi'(t')}{\varphi(t)}, \tag{E.1}$$

where $\varphi(t)$ is an increasing function of t. A short calculation gives

$$\begin{aligned} G_0^2(t,t') &= g_0^2\int_{t'}^{t}\frac{\varphi'(t'')}{\varphi(t)}\frac{\varphi'(t')}{\varphi(t'')}\,\mathrm{d}t'' \\ &= g_0^2\ln\left(\frac{\varphi(t)}{\varphi(t')}\right)\frac{\varphi'(t')}{\varphi(t)}. \end{aligned}$$

For general n, we obtain the relation

$$G_0^n(t,t') = \frac{(-1)^n}{(n-1)!}g_0^n\left(\ln\frac{\varphi(t')}{\varphi(t)}\right)^{n-1}\frac{\varphi'(t')}{\varphi(t)}. \tag{E.2}$$

Using this expression, we have

$$\frac{G_0}{1+aG_0}(t,t') = g_0\frac{\varphi^b(t')}{\varphi^b(t)}\frac{\varphi'(t')}{\varphi(t)} \tag{E.3}$$

where $b = ag_0$. For large enough t, the t'-integral of the right-hand side becomes

$$\int_0^t\frac{G_0}{1+aG_0}(t,t')\,\mathrm{d}t' = \frac{g_0}{1+ag_0}, \tag{E.4}$$

by assuming $\varphi(0)/\varphi(t) \sim 0$.

References

[1] Young A P (ed) 1997 *Spin Glass Theory and Random Fields* (Singapore: World Scientific)
[2] Barrat J L *et al* (ed) 2002 *Slow Relaxations and Nonequilibrium Dynamics in Condensed Matter* (Les Houches: Springer)
[3] Sompolinsky H and Zippelius A 1982 *Phys. Rev.* B **25** 6860
[4] Crisanti A, Horner H and Sommers H J 1993 *Z. Phys.* B **92** 257
[5] Horner H 1992 *Z. Phys.* B **86** 291
[6] Cugliandolo L F and Kurchan J 1993 *Phys. Rev. Lett.* **71** 173
[7] Cugliandolo L F and Kurchan J 1995 *Phil. Mag.* B **71** 501
[8] Mézard M, Parisi G and Virasoro M A 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
[9] Franz S, Mezard M, Parisi G and Peliti L 1998 *Phys. Rev. Lett.* **81** 1758
[10] Glauber R J 1963 *J. Math. Phys., NY* **4** 294
[11] Marinari E, Parisi G and Ritort F 1994 *J. Phys. A: Math. Gen.* **27** 7647
[12] Marinari E, Parisi G and Ritort F 1994 *J. Phys. A: Math. Gen.* **27** 7615
[13] Parisi G and Potters M 1995 *J. Phys. A: Math. Gen.* **28** 5267
[14] Nokura K 1998 *J. Phys. A: Math. Gen.* **31** 7447
[15] Nokura K 2002 *J. Phys. A: Math. Gen.* **35** 4973
[16] Nokura K 2002 *J. Phys. A: Math. Gen.* **35** 8013
[17] Sherrigton D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792
[18] Kadanoff L P and Swift J 1968 *Phys. Rev.* **165** 310
[19] Bausch R, Janssen H K and Wagner H 1976 *Z. Phys.* B **24** 113
[20] Edwards S F and Anderson P W 1975 *J. Phys. F: Met. Phys.* **5** 965
[21] de Almeida J R L and Thouless D J 1978 *J. Phys. A: Math. Gen.* **11** 983